UNIVERSITY OF MANITOBA

DATE: April 17, 2014 FINAL EXAMINATION
PAPER # 450 TITLE PAGE
EXAMINATION: Linear Algebra 2 TIME: 2 hours
COURSE: MATH 2300 EXAMINER: G.I. Moghaddam

NAME: (Print in ink) ______________________________

STUDENT NUMBER: ______________

SEAT NUMBER: ______________

SIGNATURE: (in ink) ______________________________

(I understand that cheating is a serious offense)

INSTRUCTIONS TO STUDENTS:

This is a 2 hour exam. **Please show your work clearly.**

No texts, notes, or other aids are permitted. There are no calculators, cellphones or electronic translators permitted.

This exam has a title page, 10 pages of questions and also 2 blank pages for rough work. Please check that you have all the pages. You may remove the blank pages if you want, but be careful not to loosen the staple.

The value of each question is indicated in the left hand margin beside the statement of the question. The total value of all questions is 90 points.

Answer all questions on the exam paper in the space provided beneath the question. If you need more room, you may continue your work on the reverse side of the page, but CLEARLY INDICATE that your work is continued.

Question	Points	Score
1	10	
2	9	
3	14	
4	8	
5	11	
6	8	
7	6	
8	10	
9	14	
Total:	90	

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[10] 1. For each of the following statements determine whether it is True or False.

(a) Two subsets of a vector space V that span the same subspace of V must be equal.

(b) Every linearly independent set of vectors in $\mathbf{R^n}$ is contained in some basis for $\mathbf{R^n}$.

(c) In an inner product space, if $\|\mathbf{u}\| = \|\mathbf{v}\|$, then $\mathbf{u} = \mathbf{v}$.

(d) In an inner product space, if $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, then $|\langle \mathbf{u}, \mathbf{v} \rangle| = \|\mathbf{u}\| \, \|\mathbf{v}\|$.

(e) If $\mathbf{v_0}$ is a nonzero vector in a vector space V, then the formula $T(\mathbf{v}) = \mathbf{v_0} + \mathbf{v}$ defines a linear operator on V.

(f) If $T_1 : U \to V$ and $T_2 : V \to W$ are linear transformations, and if T_1 is one-to-one, then $T_2 \circ T_1$ is also one-to-one.

(g) If A is diagonalizable, then there is a *unique* matrix P such that $P^{-1}AP$ is diagonal.

(h) If $T : V \to V$ is a linear operator, then $det(T) = det([T]_B)$ where B is any basis for V.

(i) Two similar matrices A and B have same rank, nullity and eigenvalues.

(j) If $T_1 : U \to V$ and $T_2 : V \to W$ are linear transformations, and if B, B'', and B' are bases for U, V, and W, respectively, then $[T_2 \circ T_1]_{B',B} = [T_2]_{B',B''} \, [T_1]_{B'',B}$.

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[9] 2. Let $W = \left\{ \begin{bmatrix} a & 0 \\ b & a+b \end{bmatrix} \;\middle|\; a, b \in \mathbf{R} \right\}$ be a subset of the vector space M_{22}.

(a) Prove that W is a subspace of M_{22}.

(b) Find a basis and the dimension of W.

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[14] 3. Let $A = \begin{pmatrix} -1 & 0 & -4 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$.

(a) Find all eigenvalues of A.

(b) Find bases for all eigenspaces of A.

(c) Find a matrix P that diagonalizes A.

(d) Find A^{100}. (**Hint**: You do not have to find P^{-1}).

[8] 4. Consider $\mathbf{R^5}$ with Euclidean inner product and let W be the subspace of $\mathbf{R^5}$ spanned by the vectors $\mathbf{w_1} = (1, -4, 0, 0, 0)$, $\mathbf{w_2} = (0, 0, 1, -2, 5)$ $\mathbf{w_3} = (1, -4, 0, 0, 1)$ and $\mathbf{w_4} = (0, 0, 0, 0, 1)$. Find a basis for the orthogonal complement of W.

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[11] 5. Let $W = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \;\middle|\; a, b \in \mathbf{R} \right\}$ be a subspace of the vector spaces M_{22} and let $T : P_1 \to W$ such that $T(a + bx) = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$.

(a) Show that T is a linear transformation.

(b) Find $ker(T)$.

(c) Is T an isomorphism ? Why?

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[8] 6. Let T_1 and T_2 be linear operators from $\mathbf{R}^2$ to $\mathbf{R}^2$ such that T_1 rotates a vector 90 degrees counterclockwise, and $T_2(x, y) = (x - 2y\,,\, x - y)$.

(a) Find a formula for the composition $(T_2 \circ T_1)(x, y)$.

(b) Is $(T_2 \circ T_1)^{-1} = T_1 \circ T_2$? Why?

[6] 7. Prove that if A is an $n \times n$ diagonalizable matrix with n real eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$, then

$$|A| = \lambda_1 \lambda_2 \cdots \lambda_n .$$

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[10] 8. In the vector space P_2, define the inner product of $\mathbf{p}$ and $\mathbf{q}$ by

$$\langle \mathbf{p}, \mathbf{q} \rangle = \int_0^1 p(x)q(x)dx\,.$$

Use the Gram-Schmidt process to find an orthonormal basis for $\mathbf{P}_2$ starting with the basis $\{\, \mathbf{p_1} = 2\,,\ \mathbf{p_2} = 1 - x\,,\ \mathbf{p_3} = x^2\,\}$.

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[14] 9. Let $T : P_1 \to P_1$ be a linear operator defined by $T(a+bx) = a + b(x+1)$. Also let $B = \{\mathbf{p_1} = -1\,,\ \mathbf{p_2} = x\}$ and $B' = \{\mathbf{q_1} = 2 - x\,,\ \mathbf{q_2} = 1 + x\}$ be bases for the vector space P_1.

(a) Find the matrix for T relative to the basis B.

(b) Find the matrix for T relative to the bases B and B'.

(c) Find $[T(a+bx)]_{B'}$ where $a+bx$ is any polynomial in P_1.

(d) Find the determinant, rank and nullity of T.

Answers

Q1: F T F F F F F T T T

Q2:

(a) Show that it is nonempty and closed under addition and scalar multiplication.

(b) $\{\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}0&0\\1&1\end{bmatrix}\}$ is a basis and $dim(W) = 2$.

Q3:

(a) $-1, -1, 1$.

(b) $\{(-2,1,1)\}$ is a basis for the eigenspace corresponding to $\lambda = 1$ and $\{(1,0,0), (0,1,0)\}$ is a basis for the eigenspace corresponding to $\lambda = -1$.

(c) $P = \begin{bmatrix}-2&1&0\\1&0&1\\1&0&0\end{bmatrix}$.

(d) $A^{100} = I$.

Q4: $\left\{ \begin{bmatrix}4\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\2\\1\\0\end{bmatrix} \right\}$ is a basis for $W^{\perp}$.

Q5:

(a) Show that $T(\mathbf{p}+\mathbf{q}) = T(\mathbf{p}) + T(\mathbf{q})$ and $T(k\mathbf{p}) = kT(\mathbf{p})$.

(b) $ker(T) = \{\mathbf{0}\}$.

(c) Yes because T is a linear transformation which is one to one and onto.

Q6:

(a) $(T_2 \circ T_1)(x,y) = (-2x - y\,,\, -x - y)$.

(b) Yes.

Q7: See solution of assignment 3.

Q8: $\{r_1(x) = 1\,,\, r_2(x) = \sqrt{3} - 2\sqrt{3}\,x\,,\, r_3(x) = 6\sqrt{5}\,x^2 - 6\sqrt{5}\,x + \sqrt{5}\}$ is an orthonormal basis.

Q9:

(a) $[T]_B = \begin{bmatrix}1&-1\\0&1\end{bmatrix}$.

(b) $[T]_{B',B} = \begin{bmatrix}\frac{-1}{3}&0\\\frac{-1}{3}&1\end{bmatrix}$.

(c) $[T(a+bx)]_{B'} = \begin{bmatrix}\frac{1}{3}a\\\frac{1}{3}a+b\end{bmatrix}$.

(d) $det(T) = 1$, $rank(T) = 2$ and $nullity(T) = 0$.